BIJECTIONS AND COUNTING OF DECORATED COMBINATORIAL MAPS

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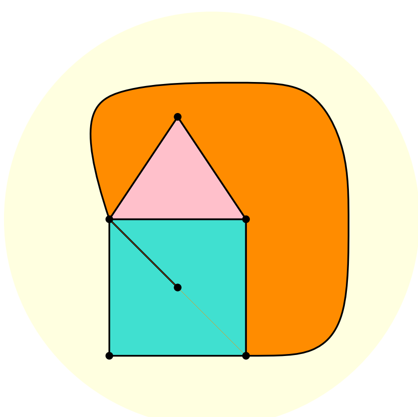
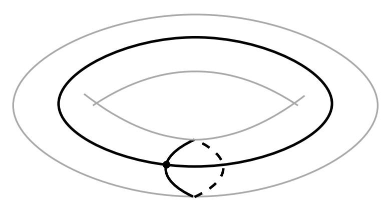
Context

Combinatorial maps are graphs that are embedded on surfaces and as such they have more structure than graphs [MT01]. Maps have appeared in numerous areas of computer science (mostly combinatorics of course), but also in mathematics and in physics. The enumeration of maps in particular has revealed many fascinating properties. It started with Tutte's pioneering work in the 60s, then with physicists in the 80s and the subject was revitalized in the 00s with the advent of new bijections like Schaeffer's bijection with trees. Meanwhile the original approach of Tutte, based on functional equations for the generating functions is still a very active topic [BM11], in relation to formal computing for instance.



(A) A planar map where we colored the complement of the graph.



(B) A map with two edges and one vertex on the torus.

Recently the interest has shifted towards the enumeration of maps that are decorated. For instance, maps decorated with loops are counted also with respect to the number of loops [BBG11]. Objects such as loops affect the large scale limit and the universality class because they are non-local and can be arbitrarily large. However, obtaining these kinds of results (e.g. exact or asymptotic enumeration) is very challenging for these models. For instance, the functional equations are more difficult. They may involve two so-called catalytic variables instead of just one. They may require elliptic parametrization instead of rational

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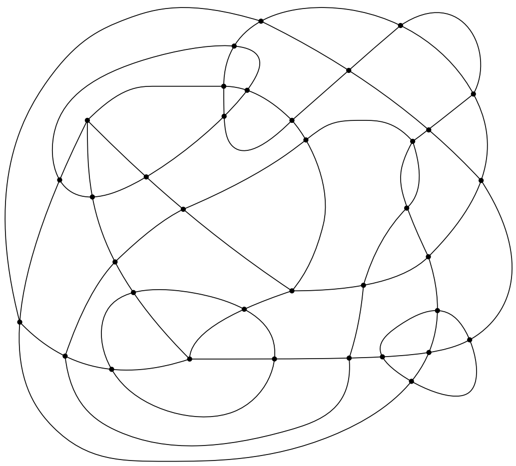
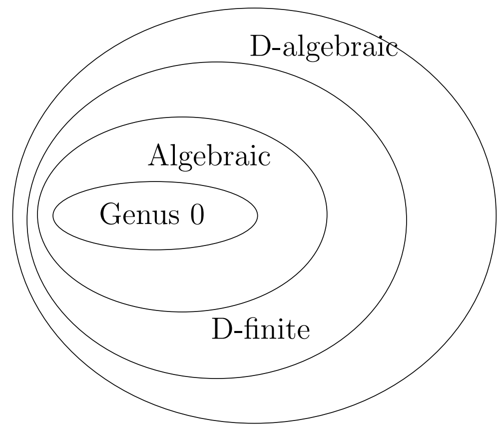
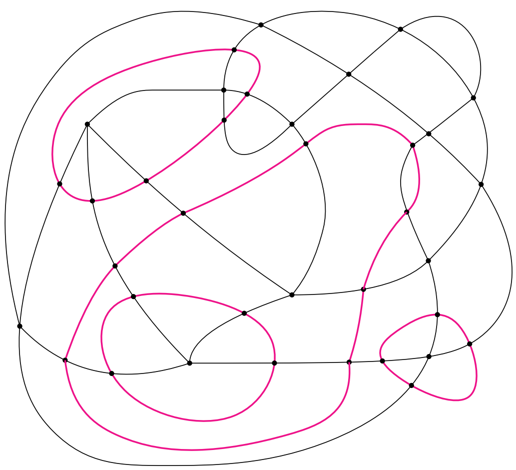
ones [EZJ23]. As a result, the generating function can be non-algebraic (e.g. D-finite). Moreover, and not unrelated, very few bijections exist in those cases.



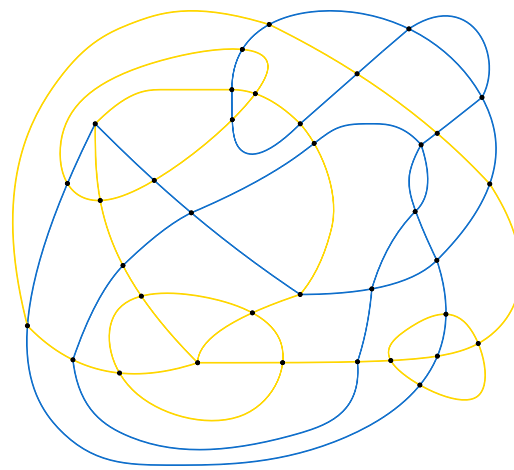
(A) A 4-regular planar map.



(B) A hierarchy of families of generating functions.



(C) A 4-regular planar map decorated with loops.



(D) A 4-regular planar map with an ABAB configuration. It has vertices of 3 types: incident to either only blue edges, or only gold edges, or to two blue and two gold alternating edges.

The project

The student will learn the topic of solving the functional equations of maps (the so-called differential-catalytic equations) à la Tutte, and the basic bijections of the field. The student will then consider one or more models of decorated maps. Some interesting ones are: the ABAB model solved by physicists but not understood from functional equations and bijections; the 8-vertex model which is a generalization of the 6-vertex model solved in [EZJ23] coming statistical mechanics; the Gessel walks which lie in the same universality class as critical Ising-decorated maps but for which no bijections are known [BM16].

Research environment: LIGM

The supervisor is a member of the COMBI team in LIGM, Université Gustave Eiffel (formerly known as Paris-Est Marne), which has a focus on enumerative, bijective and algebraic combinatorics. The lab has a strong position in combinatorics, and at least three teams involved in some developments on maps. The student would thus benefit from a rich environment, including multiple experts on maps.

References

[BBG11] G. Borot, J. Bouttier, and E. Guitter. A recursive approach to the o(n) model on random maps via nested loops. *Journal of Physics A: Mathematical and Theoretical*, 45(4):045002, dec 2011.

[BM11] Mireille Bousquet-Mélou. *Counting planar maps, coloured or uncoloured*, page 1–50. London Mathematical Society Lecture Note Series. Cambridge University Press, 2011.

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[EZJ23] Andrew Elvey Price and Paul Zinn-Justin. The six-vertex model on random planar maps revisited. *Journal of Combinatorial Theory, Series A*, 196:105739, 2023.

[MT01] Bojan Mohar and Carsten Thomassen. *Graphs on surfaces*, volume 2. Johns Hopkins University Press Baltimore, 2001.